MATHEMATICAL EXCURSIONS AND PASTIMES

G.S. BADERIA

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0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

x	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

MATHEMATICAL EXCURSIONS
and
PASTIMES

Book I

Understanding Modern Mathematics Series

Mathematical Excursions

AND

Pastimes

BOOK I

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Preface

Ever since the dawn of civilization, man has been obsessed by his possessive instinct. He began possessing herd of cattle and farmland. In this process, he needed to count and measure. This gave rise to the science of numbers and space. He called it mathematics.

Today, mathematics is no longer a study of numbers and spatial relations alone. It has gone far beyond this. Today mathematics is a study of structures. The elements and operations giving rise to these structures need not be numbers or the familiar operations of addition and multiplication.

In spite of the growing abstractness of mathematics, applications of mathematics have been growing. Today the writ of mathematics runs in physical, biological, social and management sciences, commerce, trade and industry, psychology, etc.

In addition to these practical and intellectual values, mathematics has its appealing recreational aspects. Many mathematical principles can be applied in amusing ways. Applying these principles in this manner brings much personal enjoyment.

Although there is no dearth of literature on this recreational aspect of mathematics, there is generally no attempt to explain the principles on which such problems are based. In the present book an effort has been made to take into consideration this view-point, so that the reader has also an insight into the "why" of all the "tricks" that work or the manner in which problems are solved.

The book has been written for use by mathematics club sponsors, science talent search contestants, teacher training schools and colleges and by those who are interested in mathematics and have some leisure time to spare.

I am grateful to my friend Sq. Leader Mr. Garde Instructor or Air Force Staff Training College who read the entire text at the manuscript stage and made many valuable suggestions.

Any suggestions received from readers will be greatly appreciated.

G. S. Baderia

March 1, 1975

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1

Magic Squares

1·1. In several Indian homes, especially in the country side, you may come across squares containing nine, sixteen or twenty five boxes with numbers written in those boxes in such a manner that the sum of numbers in each row, in each column and diagonally is the same. Such squares have been given the name "magic squares". We shall discuss here properties of magic squares and develop rules to construct them.

Here is a magic square containing nine boxes,

(*i*) Nine consecutive numbers from 1 to 9 have been put in these nine boxes.

(*ii*) Column-wise, row-wise and diagonally the sum of numbers is 15. This constant sum is called the "magic constant" for a square.

(*iii*) The central box contains 5.

6	1	8
7	5	3
2	9	4

2	9	4
7	5	3
6	1	8

(*iv*) The above characteristics of a magic square would not be lost, if we were to reflect a row (or column) into the other with respect to the central row (or column). Observe the magic square obtained by reflecting first row into the third and *vice-versa*.

Similarly, we have a column-reflection of the above magic square as below :

8	1	6
3	5	7
4	9	2

We may also note that the sum of each row or column or diagonal is 15 which is $\frac{1}{3}$ the total sum of the numbers used. This is so because $1+2......+9=\frac{9(1+9)}{2}=45.$

Moreover, these numbers are put in three rows (or columns).

Each row or column has the same sum, so that it would be $\frac{1}{3}\times 45=15.$

(*v*) Number 5 in the central box happens to be the mean of all the numbers used. It is also the mean of each row, column and diagonal.

The number n of rows (or of columns) of a square is called its *order*. Thus a 3×3 square is a square of the third order. A 4×4 square is a square of the fourth order.

In this chapter, we shall discuss construction of magic squares of order three, four, five, six and eight.

Problem 1. *Construct a 3×3 magic square (square of the third order) such that the sum of each row, column and diagonal is 36.*

Discussion. The mean of each row is $36\div3=12.$

We, therefore, need nine consecutive numbers namely 8, 9, 10, ..., 15, 16.

Now we have $8+12+16=36$

$$9+12+15=36$$

$$10+12+14=36$$

and $11+12+13=36$

We shall take (8, 12, 16) or (10, 12, 14) for the middle row. (Why ?)*

---

*We have two odd number triples and two even number triples, such that 12 is their common element. We can arrange the even number triples either diagonally as shown in the following figure :

8		10
	12	
14		16

	8	
14	12	10
	16	

or we can arrange them vertically and horizontally as shown in the next figure.

In the first case, we again need even numbers to ensure that column-wise and row-wise sum is 36 which is even. This is not possible. Hence we have only the second alternative left.

For 16, the other two numbers are 9, 11 so as to make a total of 36. *i.e.*, $9+16+11=36$.

8	12	16

		9
8	12	16
		11

After putting 9 and 11 as shown in the adjoining figure, we turn to the diagonal containing 9, 12 and we have $9+12+15=36$.

		9
8	12	16
15		11

		9
8	12	16
15	10	11

The first column must have 13 in the first box so that $13+8+15=36$ and the middle box in the third row must have 10. It is now not difficult to get numbers 13 and 14 for the remaining two boxes.

13	14	9
8	12	16
15	10	11

We may now check :

$13+14+9=36$ $\quad$ $13+8+15=36$ $\quad$ $13+12+11=36$

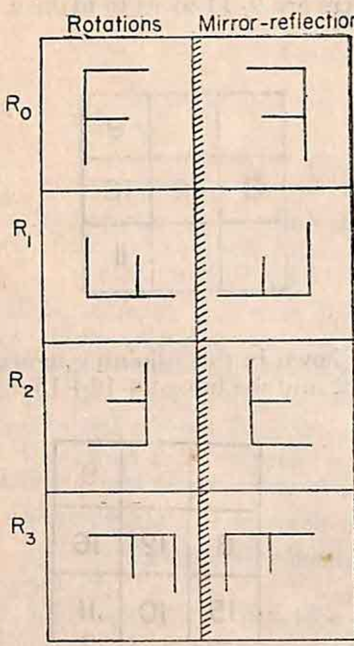
$8+12+16=36$ $\quad$ $14+12+10=36$ and $15+12+9=36$

$15+10+11=36$ $\quad$ $9+16+11=36$;

Hence the magic square is properly constructed.

1·2. Transformation of squares of an odd order.

Magic squares which have odd number of boxes in a row (or column) are called squares of an odd order or odd box squares. An odd-box magic square may be rotated through a right angle and may be reflected without changing the characteristics of magic square. There are eight such transformations of a magic square as illustrated by the following :

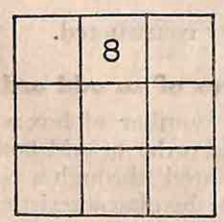


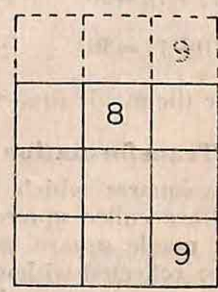
	13	14	9	9	14	13
M (R_0)	8	12	16	16	12	8
	15	10	11	11	10	15
	9	16	11	11	16	9
M (R_1)	14	12	10	10	12	14
	13	8	15	15	8	13
	11	10	15	15	10	11
M (R_2)	16	12	8	8	12	16
	9	14	13	13	14	9
	15	8	13	13	8	15
M (R_3)	10	12	14	14	12	10
	11	16	9	9	16	11

Each of these eight magic squares are considered equivalent and as such are not considered disinct.

One of the methods of constructing nine-box magic squares has been given above. We shall give below a general method of constructing odd-box squares, illustrating it with a 3×3 square.

(1) Draw a nine-box blank square. Insert the smallest number of the series of consecutive numbers proposed to be used for constructing the magic square. Suppose we went to fill in a 3×3 square with numbers 8, 9,..., 16. Then insert 8 in the central box of the first row.





(2) Insert the next consecutive number *i.e.*, 9 one box to the

right and one box up. For this, consider the magic square rolled up on a cylinder so that bottom row becomes adjacent to the top row as shown on the preceding page. Similarly consider the left column as adjacent to the right column. 9 written in dotted box thus belongs to the bottom right hand box.

	8		
10			10
		9	

(3) Next number 10 will be inserted in the dotted box as shown here and it really belongs to the central box in the left column.

	8	13
10	12	
11		9

(4) Next number 11 belongs to a column which is already occupied by 8. In such situations, put the next number in the box immediately below as shown here.

(5) It is easy to fill in next two boxes by 12 and 13.

(6) Next number 14 cannot be placed because it falls in a diagonal position where there is no continuation of a column or row. In this case also, we insert the number in the box below that occupied by 13.

(7) 15 and 16 are filled as shown in the adjoining figure.

	16		
15	8	13	15
10	12	14	
11	16	9	

Let us use these rules to fill in a 25-box square with numbers from 1 to 25, as shown below :

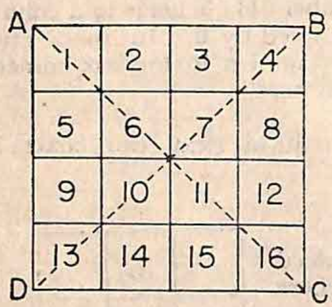
	18	25	2	9	
17	24	1	8	15	17
23	5	7	14	16	23
4	6	13	20	22	4
10	12	19	21	3	10
11	18	25	2	9	

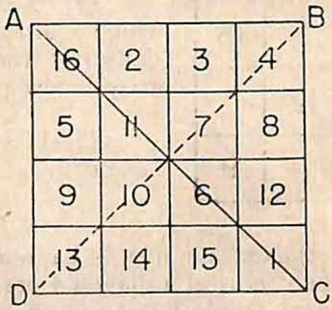
This magic square of 5*th* order using numbers 1 to 25 will be referred to as the standard magic square of the 5*th* order.

1.3. Even-box magic squares (*i.e.*, squares of even order).

4–4 Square—The method for constructing a 4×4 magic square is different from that described above for an odd-box magic square. In this we proceed as below :

(1) Draw a 4×4 blank square. Suppose we want to construct this square with the help of numbers from 1 to 16, then fill in these numbers as shown in the first figure given below :





(2) Now reflect the numbers of one set of diagonal boxes with respect to the other diagonal, *e.g.*, reflect the numbers 1, 6, 11 and 16 contained in boxes intersected by diagonal AC with respect to diagonal BD. We have the transformed square as shown above.

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

(3) Next reflect numbers 4, 7, 10 and 13 with respect to diagonal AC. We have the desired square as shown here.

The total of each row, column and diagonal is 34.

This is referred to as the standard magic square of the 4*th* order.

	2	3	
5			8
9			12
	14	15	

With a little practice, we can perform the desired reflection simultaneously as below :

(1) Leave out the numbers meant for diagonal boxes and insert the remaining as shown here.

(2) Start filling in the remaining numbers, *i.e.*, 1, 4, 6, 7, 10, 11, 13, 16 in the blank boxes beginning from the right bottom and moving from right to left as shown in the adjoining square.

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

6×6 square. Six by six square has to be regarded as a union of 4 nine-box squares. It has to be filled in as follows :

(1) Fill in the top-left 3×3 square by numbers 1 to 9, and bottom-right 3×3 square by numbers 10 to 18.

8	1	6			
3	5	7			
4	9	2			
			17	10	15
			12	14	16
			13	18	11

(2) Next fill in the top-right square by numbers 19 to 27, and bottom-left square by numbers 28 to 36 as shown in the adjoining figure.

8	1	6	26	19	24
3	5	7	21	23	25
4	9	2	22	27	20
35	28	33	17	10	15
30	32	34	12	14	16
31	36	29	13	18	11

(3) Next transpose numbers intersected by segment chain into similarly placed boxes, *i.e.*, the numbers (8, 35), (5, 32) and (4, 31) will change places and the final square will appear as given here.

35	1	6	26	19	24
3	32	7	21	23	25
31	9	2	22	27	20
8	28	33	17	10	15
30	5	34	12	14	16
4	36	29	13	18	11

This is referred to as the standard magic square of the 6*th* order and its magic constant is 111.

8 × 8 Square—Eight by eight square is constructed as 4 × 4 square leaving out diagonal boxes blank. For considering the diagonal boxes the entire square should be considered as union of four by four squares.

	2	3			6	7	
9			12	13			16
17			20	21			24
	26	27			30	31	
	34	35			38	39	
41			44	45			48
49			52	53			56
	58	59			62	63	

Next fill in the vacant boxes starting from bottom right box and beginning from 1 onwards till 64. Numbers should be left for boxes which are not intersected by diagonals.

64			61	60			57
	55	54			51	50	
	47	46			43	42	
40			37	36			38
32			29	28			25
	23	22			19	18	
	15	14			11	10	
8			5	4			1

Now superpose the two squares and you have constructed a 8 × 8 magic square.

Row-wise, column-wise and diagonally, the total is 260.

64	2	3	61	60	6	7	57
9	55	54	12	13	51	50	16
17	47	46	20	21	43	42	24
40	26	27	37	36	30	31	33
32	34	35	29	28	38	39	25
41	23	22	44	45	19	18	48
49	15	14	52	53	11	10	56
8	58	59	5	4	62	63	1

1·4. Some more properties of magic squares.

(1) If a constant number be added to each number in a given magic square, the resulting square retains the property of magic squares. *e.g.*,

8	1	6
3	5	7
4	9	2

Add a number c (say c=5) to each number $\longrightarrow$

8+5	1+5	6+5
3+5	5+5	7+5
4+5	9+5	2+5

=

13	6	11
8	10	12
9	14	7

Row and column-wise (and diagonally also) total is 15 in the first square. It is easily seen that the total in the next square is $15+3\times C=15+3\times 5=30$ [since C=5 in the above example].

The set of consecutive numbers used for the subsequent square is {6, 7, 8, ...14}.

(2) If a constant number be subtracted from each number in a given magic sauare, the resulting square retains the property of magic square, *e.g.*,

8	1	6
3	5	7
4	9	2

Subtract a number say 5, from each number $\longrightarrow$

8−5	1−5	6−5
3−5	5−5	7−5
4−5	9−5	2−5

=

3	−4	1
−2	0	2
−1	4	−3

Row, column and diagonal wise, the total in the resulting square is "zero".

The set of consecutive numbers used is $\{-4, -3, \ldots, 0, 1, \ldots, 4\}$

(3) If each number of a magic square is multiplied by a constant, the resulting square is also magic, *e.g.*,

8	1	6
3	5	7
4	9	2

Multiply each number $\longrightarrow$ by a constant(say6)

8X6	1X6	6X6
3X6	5X6	7X6
4X6	9X6	2X6

=

48	6	36
18	30	42
24	54	12

Row, column and diagonal wise, the total in the resulting square is $15\times 6=90$.

(4) If each number of a magic square is divided by a constant, the resulting square is also magic, *i.e.*

8	1	6
3	5	7
4	9	2

$\xrightarrow[\text{by a constant (say 3)}]{\text{Divide each number}}$

8÷3	1÷3	6÷3
3÷3	5÷3	7÷3
4÷3	9÷3	2÷3

8/3	1/3	2
1	5/3	7/3
4/3	3	2/3

Row, column and diagonal wise, the total in the resulting square is $15 \div 3 = 5$.

Problem 2. *Complete a magic square of the third order given a number in a particular cell.*

Campare the given square with the standard square using 1—9

11		

8	1	6
3	5	7
4	9	2

We have the following possibilities :

$$11 = 8 + 3$$
$$11 = 8 \div 2 + 7$$
$$11 = 8 \times 2 - 5$$
$$11 = 8 \times \tfrac{11}{8}$$

and we can construct the following four squares,

(*i*)

8+3	1+3	6+3
3+3	5+3	7+3
4+3	9+3	2+3

=

11	4	9
6	8	10
7	12	5

(*ii*)

8×2 −5	1×2 −5	6×2 −5
3×2 −5	5×2 −5	7×2 −5
4×2 −5	9×2 −5	2×2 −5

=

11	−3	7
1	5	9
3	13	−1

(iii)

$8 \div 2 + 7$	$1 \div 2 + 7$	$6 \div 2 + 7$
$3 \div 2 + 7$	$5 \div 2 + 7$	$7 \div 2 + 7$
$4 \div 2 + 7$	$9 \div 2 + 7$	$2 \div 2 + 7$

=

11	$15/2$	10
$17/12$	$19/2$	$21/2$
9	$23/2$	8.

(iv)

$8 \times \frac{11}{8}$	$1 \times \frac{11}{8}$	$6 \times \frac{11}{8}$
$3 \times \frac{11}{8}$	$5 \times \frac{11}{8}$	$7 \times \frac{11}{8}$
$4 \times \frac{11}{8}$	$9 \times \frac{11}{8}$	$2 \times \frac{11}{8}$

=

11	$\frac{11}{8}$	$\frac{33}{4}$
$\frac{33}{8}$	$\frac{55}{8}$	$\frac{77}{8}$
$\frac{11}{2}$	$\frac{99}{8}$	$\frac{11}{4}$

In the first solution, the magic constant is 24 ;

In the second solution, the magic constant is 15 ;

In the third solution, the magic constant is $\frac{57}{2}$;

In the fourth solution, the magic constant is $\frac{165}{8}$;

If we want to avoid negative and fractional solutions, then the first square is the desired solution.

Problem 3. *Complete the following magic square of the third order which has a magic constant of 30.*

13		

Since the magic constant is 30, the central box must have $30 \div 3 = 10$.

Now compare it with the standard magic square. Since we can multiply each number of magic square by a constant number c and add another constant number d,

We have $13 = 8 \times c + d$
and $10 = 5 \times c + d$

13		
	10	

8	1	6
3	5	7
4	9	2

We have two independent equations and two variables and we can solve them uniquely so that $c=1$ and $d=5$.

We may therefore, complete the square as given below :

8+5	1+5	6+5
3+5	5+5	7+5
4+5	9+5	2+5

=

13	6	11
8	10	12
9	14	7

Problem 4. *Complete the magic square of fourth order given* $x_{1,1}=40$ *and* $x_{3,4}=32$ *where* $x_{i,j}$ *stands for an entry in i th row and j th column.*

Compare the given entries with those in the standard square.

40			
			32

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

Since we can multiply each number of magic square by a constant number c and add another constant number d, we have

$$\begin{cases} 16 \times c + d = 40 \\ 12 \times c + d = 32 \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} c = 2 \\ d = 8 \end{cases}$$

Hence the magic square is :

16X2 +8	2X2 +8	3X2 +8	13X2 +8		40	12	14	34
5X2 +8	11X2 +8	10X2 +8	8X2 +8	=	18	30	28	24
9X2 +8	7X2 +8	6X2 +8	12X2 +8		26	22	20	32
4X2 +8	14X2 +8	15X2 +8	1X2 +8		16	34	38	10

The numbers used are consecutive even numbers from 10 to 48. The magic constant is 100.

Problem set 1·1

1. Construct a magic square of the *3rd* order using the following positive integers :
   (*i*) { 1, 2, ..., 8, 9 }
   (*ii*) { 5, 6, ..., 13 }
   (*iii*) { 8, 9, ..., 16 }
   (*iv*) { 11, 12, ..., 19 }
2. Construct a magic square of the *4th* order using the following positive integers :
   (*i*) { 1, 2, ..., 16 }
   (*ii*) { 3, 4, .. , 18 }
   (*iii*) { 11, 12, .. , 26 }
   (*iv*) { 51, 52, . , 66 }
3. Construct a magic square of the *5th* order using the following positive integers :
   (*i*) { 1, 2, ..., 25 }
   (*ii*) { 6, 7, ..., 30 }
   (*iii*) { 11, 12, ..., 35 }
   (*iv*) { 21, 22, ..., 45 }
4. Construct the following magic squares :
   (*i*) Magic square of the *6th* order using numbers in the set { 1, 2, ..., 36 }
   (*ii*) Magic square of the *6th* order using numbers in the set { 6, 7, , 41 }

(*iii*) Magic square of the 8*th* order using numbers in the set $\{17, 18, \ldots, 80\}$

(*iv*) Magic square of the 8*th* order using numbers in the set $\{11, 12, \ldots, 74\}$

5. Construct a magic square of the third order such that

(*i*) the magic constant is 21

(*ii*) the magic constant is 45

(*iii*) the magic constant is 51

(*iv*) the magic constant is 150

6. Construct a magic square of the fourth order such that

(*i*) the magic constant is 42

(*ii*) the magic constant is 50

(*iii*) the magic constant is 68

(*iv*) the magic constant is 340

7. Construct a magic square of the 5*th* order such that

(*i*) the magic constant is 70

(*ii*) the magic constant is 130

(*iii*) the magic constant is 140

(*iv*) the magic constant is 650

8. Complete a magic square of third order if

(*i*) $x_{1,1}=10$

(*ii*) $x_{2,3}=12$

(*iii*) $x_{3,3}=9$

$x_{i,j}$ refers to the number in the ith row and jth column. How many solutions are possible in each case ?

9. Complete a magic square of the third order such that

(*i*) $x_{1,1}=10$, $x_{1,2}=3$

(*ii*) $x_{1,1}=10$, $x_{1,2}=-4$

(*iii*) $x_{1,1}=24$, $x_{1,2}=3$

10. Complete a magic square of the fourth order such that

(*i*) $x_{1,2}=4$, $x_{2,3}=12$

(*ii*) $x_{1,2}=4$, $x_{2,3}=20$

(*iii*) $x_{1,2}=-4$ $x_{2,3}=12$

11. Complete a magic square of the fifth order such that $x_{2,\,2}=24$ and magic constant$=264$

[**Hint**—Compare with the standard magic square of the fifth order so that

$$\begin{cases} 5\times c+d=24 \\ 65\times c+d=264 \end{cases} \Leftrightarrow \begin{cases} c=4 \\ d=4 \end{cases}$$

Construct the magic square multiplying each number of the standard magic square by 4 and adding it by 4.]

12. Construct the following magic squares :

(*i*) Magic square of the third order such that $x_{2,\,3}=20$ and magic constant$=36$

(*ii*) Magic square of the fourth order such that $x_{3,\,4}=28$ and magic constant$=72$

(*iii*) Magic square of the fifth order such that $x_{4,\,5}=8$ and magic constant$=132$

(*iv*) Magic square of the sixth order such that $x_{1,\,6}=50$ and magic constant$=224$

(*v*) Magic square of the eighth order such that $x_{8,\,8}=3$ and magic constant$=780$

2

Divisibility of Numbers

2.1. Ever since I was a young boy, I was fascinated by an interesting property shown by numbers in the multiplication table of 9, that if we add up the digits of any such number (multiple of 9), the sum would be 9 or a multiple of 9.

$9\times1=9$		
$9\times2=18$	;	$1+8=9$
$9\times8=72$	;	$7+2=9$
$9\times12=108$	;	$1+0+8=9$
$9\times15=135$	;	$1+3+5=9$
$9\times28=252$	;	$2+5+2=9$
$9\times65=585$	;	$5+8+5=18$
$\vdots$		$\vdots$

Since the multiplication table for 9 produces multiples of 9, we may restate the problem as below :

"If a number is divisible by 9, then the sum of its digits is also divisible by 9."

"Conversely if we are given any number, we can immediately state by inspection if it is not divisible by 9." For example 341256399 is not divisible by 9. In fact it leaves a remainder of **6** on division by 9.

A number of rules have been developed to find out by inspection if a given number is divisible by another number. (Of course, these rules have been made only for a few divisors). Instead of developing these rules by varying procedures, we will here discuss the concept of congruence of numbers and then use it to develop the rules.

2.2. Congruence of Numbers.

It often happens that for the purposes of a certain calculation, two numbers which differ by a multiple of some fixed number are considered as equivalent, in the sense that they produce the same result. For example, if a train starts at 6 a.m. from station A, arrives 10 hours thereafter to station B and 34 hours after leaving station A to station C, then the train would arrive at B and C at the same hour of a day *i.e.* 4 p.m.

Similarly if the 1st April of some year is a Sunday, then 1st July of that year will also be a Sunday. This is so because the two dates differ by 91 days and 91 is a multiple of 7, the number of days in a week.

The value of $(-1)^n$ depends on whether n is odd or even, so that two values of n which differ by a multiple of 2 give the same result. The value of $(\sqrt{-1})^n$ exhibits the following relationship :

$(\sqrt{-1})^2=-1$, $(\sqrt{-1})^3=-\sqrt{-1}$, $(\sqrt{-1})^4=1$, $(\sqrt{-1})^5=\sqrt{-1}$ and so on.

In other words, the value of $(\sqrt{-1})^n$ and $(\sqrt{-1})^m$ would be the same if n and m differ by 4 or some multiple of 4.

The congruence notation, introduced by Carl Friedrick Gauss (1777—1855), expresses in a convenient form the fact that two integers a and b differ by a multiple of a fixed natural number m. In the notation of Gauss, the fact that two integers a and b differ by some integral multiple of a natural number m is written as $a \cong b \pmod m$ and read as : a is congruent to b with respect to the modulus m.

Examples. $63 \cong 0 \pmod 7$ because $(63-0)$ is divisible by 7

$41 \cong 1 \pmod 2$ because $(41-1)$ is divisible by 2

$5^2 \cong -1 \pmod{13}$ because $(5^2-{}^-1)$ is divisible by 13

Operations on Congruences. Congruences can be added, subtracted or multiplied in just the same way as equations, provided all the congruences have the same modulus.

(*A*) *To show that if* :

$$a_1 \cong b_1 \ (mod\ m)$$

and $$a_2 \cong b_2 \ (mod\ m)$$

then (*i*) $$a_1+a_2 \cong b_1+b_2 \ (mod\ m)$$

(*ii*) $$a_1-a_2 \cong b_1-b_2 \ (mod\ m)$$

(*iii*) $$a_1.a_2 \cong b_1.b_2 \ (mod\ m)$$

$a_1 \cong b_1 \pmod m \Rightarrow a_1-b_1=q_1.m$ where $q_1 \in \mathrm{I}$

...(by definition)

$$\Leftrightarrow a_1=b_1+q_1m$$

$a_2 \cong b_2 \pmod m \Rightarrow a_2=b_2+q_2m$ where $q_2 \in \mathrm{I}$

So that $a_1+a_2=b_1+b_2+(q_1+q_2)m$

$$\cong b_1+b_2 \pmod m$$

Similarly $a_1-a_2=b_1-b_2+(q_1-q_2)m$

$\cong b_1-b_2 \pmod m$, since $q_1-q_2 \in \mathrm{I}$

and $a_1.a_2=(b_1+q_1m)(b_2+q_2m)$

$=b_1b_2+(q_1b_2+q_2b_1)m+q_1q_2m^2$

$=b_1b_2+(q_1b_2+q_2b_1+q_1q_2m)m$

$\cong b_1b_2 \pmod m$ since $q_1b_2+q_2b_1+q_1q_2m \in \mathrm{I}$

(*B*) In general, we can show that if :

$a_1\cong b_1 \pmod m$; $a_2\cong b_2 \pmod m$; $a_3\cong b_3 \pmod m$,...

Then (*i*) $a_1+a_2+a_3+\ldots\cong b_1+b_2+b_3+\ldots \pmod m$

(*ii*) $a_1-a_2-a_3-\ldots\cong b_1-b_2-b_3-\ldots \pmod m$

(*iii*) $a_1.a_2.a_3\ldots\cong b_1.b_2.b_3\ldots \pmod m$

2.3. Application of Congruence of Numbers.

(*I*) *Divisibility by 9 and 3.*

Theorem (A) and (B) listed above help us to develop a number of useful rules about divisibility. Let us consider divisibility by 9.

We know that $10\cong 1 \pmod 9$

$10^2=10\times 10\cong 1\times 1 \pmod 9$ $\quad\therefore 10\cong 1 \pmod 9$

Similarly $10^3\cong 1 \pmod 9$

In general $10^n\cong 1 \pmod 9$ where $n \in \mathrm{N}$

Now let the digits of any number, that we want to investigate be $a_0, a_1, a_2, a_3,\ldots, a_n$ beginning from units, tens etc., so that the number would be written as $a_n\ldots a_3\, a_2\, a_1\, a_0$.

[Consider $a_0, a_1,\ldots$ as numbers]

In expanded notation, the number would be

$$a_n.10^n+\ldots+a_3.10^3+a_2.10^2+a_1.10+a_0$$

Now $a_0\cong a_0 \pmod 9$

$a_1.10\cong a_1.1 \pmod 9$

$\cong a_1 \pmod 9$

$a_2.10^2\cong a_2.1 \pmod 9)\cong a_2 \pmod 9$

$\vdots \qquad \vdots$

So that $a_n.10^n+\ldots+a_3.10^3+a_2.10^2+a_1.10+a_0$

$$\cong a_n+\ldots+a_3+a_2+a_1+a_0 \pmod 9$$

Explaining the above result in words we may say :

If we divide any given number by 9, it leaves the same remainder as is left by number obtained by adding up its digits.

From this it follows :

(*i*) If a number is divisible by 9, then the sum of its digits is divisible by 9, and

(*ii*) A number is divisible by 9, if the sum of its digits is divisible by 9.

In mathematical language (*i*) and (*ii*) are combined in the following manner :

A number is divisible by 9, if and only if the sum of its digits is divisible by 9.

Problem Set 2.1

1. Proceeding in the manner as in I above, prove that :

   A number is divisible by 3, if and only if the sum of its digits is divisible by 3.

2. Find without actual division if the following numbers are divisible by 9.

   391,230 ; 456,003 ; 321,012 ; 965,433.

3. Are the numbers given in problem 2 above divisible by 3 ?

4. Which of the following statements are true ?

   (*i*) If a number is divisible by 9, it is divisible by 3.

   (*ii*) If a number is divisible by 3, it is divisible by 9.

5. If two numbers have the same digits but in opposite order, prove that their difference is always divisible by 9. Try for instance 9847 and 7489, and then try to generalise.

   Solution.

   (*i*) $9847-7489=2358$

   2358 is divisible by 9, because the sum of its digits is 18 which is divisible by 9.

(*ii*) In general consider any number with units, tens and subsequent digits $a_0, a_1, a_2, \ldots a_n$.

The sum of their digits would be $a_0+a_1+ \cdot +a_n$.

The number with the same digits but in opposite order would also have the same sum of its digits *i.e.*, $a_0+a_1+a_2+\ldots+a_n$.

By the rule for divisibility by 9, the two numbers leave the same remainder when divided by 9 (because their digits have the same sum), so that if we denote the numbers by n_1 and n_2,

then $$n_1=q_1.9+r_1$$
$$n_2=q_2.9+r_1$$

So that $n_1-n_2=(q_1-q_2)9$

6. Given some digits (say 5, 4, 7), construct all possible numbers with three digits, using each digit only once. Show that the difference of any two numbers so formed is divisible by 9.

   Take any other set of digits and verify your statement.

7. Show that the sum of any two numbers formed in problem (6) above is not divisible by 9.

   [Try 547+745 ; 574+457 ; 475+754 ; ...]

8. Show that problems 5, 6, 7 are valid for divisibility by 3 also.

(*II*) *Divisibility by 2 and 5*

You know that a number is divisible by 2 or by 5, if and only if the unit's digit in a number is zero or is divisible by 2 or 5. Let us see how we can use the idea of congruence of numbers to arrive at this rule.

Let the digits of any number be $a_0, a_1, a_2, \ldots, a_n$ (beginning from units onwards). The number in expanded notation would be

$$a_n.10^n+\ldots+a_2.10^2+a_1.10+a_0$$

Now $$10 \cong 0 \pmod 2$$

$$10^2=10\times10\cong10\times0 \pmod 2$$
$$\cong 0 \pmod 2$$

In general $$10^n \cong 0 \pmod 2$$

Hence $$a_n.10^n \cong a_n.0 \pmod 2$$
$$\cong 0 \pmod 2$$
$$\vdots \qquad \vdots$$
$$a_2.10^2 \cong 0 \pmod 2$$
$$a_1.10 \cong 0 \pmod 2$$
$$a_0 \cong a_0 \pmod 2$$

so that $a_n.10^n + \ldots + a_2.10^2 + a_1.10 + a_0 \cong a_0 \pmod 2$

Explaining the above result in words, we may say :

If we divide any given number by 2, it leaves the same remainder as is left by its digit in the unit's place.

From this it follows :

A number is divisible by 2 iff its unit's digit is divisible by 2.

Proceeding in a similar manner as above, we can show that :

A number is divisible by 5 iff its unit's digit is 0 or 5.

(III) Divisibility by 11

Consider a number 567981.

We have $567981 = 56 \times 10^4 + 79 \times 10^2 + 81$.

Now $$10^2 = 100 = 99 + 1 \cong 1 \pmod{11}$$
$$10^4 = 10^2 \times 10^2 \cong 1 \times 1 \pmod{11}$$

so that $$56{,}79{,}81 \cong 56 + 79 + 81 \pmod{11}$$
$$\cong 216 \pmod{11}$$

To test the divisibility of a number by 11, divide the number in blocks of two digits beginning from the unit's digit. Add the blocks so formed. If the sum is divisible by 11, then the number is divisible by 11.

Consider another method :

$$10 = (11 - 1) \cong -1 \pmod{11}$$
$$10^2 \cong (-1)(-1) \pmod{11}$$
$$\cong 1 \pmod{11}$$
$$10^3 \cong -1 \pmod{11}$$

In general $\quad 10^n \equiv (-1)^n \pmod{11}$

so that $\quad a_n.10^n + \ldots + a_3.10^3 + a_2.10^2 + a_1.10 + a_0$

$$\cong a_n(-1)^n + \ldots + a_3(-1)^3 + a_2(-1)^2 + a_1(-1) + a_0 \pmod{11}$$

$$\cong a_n(-1)^n + \ldots - a_3 + a_2 - a_1 + a_0 \pmod{11}$$

$$\cong (a_0 + a_2 + a_4 + a_6 + \ldots) - (a_1 + a_3 + a_5 + a_7 + \ldots) \pmod{11}$$

Explaining the above result in words, we may say :

To find out if any given number is divisible by 11, add all the digits at odd places beginning from the unit's digit then add digits at even places. If the difference of these sums is divisible by 11, then the number is also divisible by 11.

Example.

(*i*) 943,241 is not divisible by 11, because

$(1+2+4)-(4+3+9) = -9$ which is not divisible by 11

(*ii*) 3,423,398 is divisible by 11, because

$(8+3+2+3)-(9+3+4) = 0$ which is divisible by 11.

Problem Set 2.2

1. Find by inspection if the following numbers are divisible by 2 or 5 or both :

2423, 3422, 5645, 64920.

2. Find by inspection if the following numbers are divisible by 11 :

34231, 56870, 943,349.

Use both methods discussed in the preceding pages and see if you get the same result.

3. Find a rule to determine the divisibility of a number by 4.

4. Show that for any integer n, $n(n^2+1)$ is always even.

[**Hint**. n is either even or odd.

If n is even, then any multiple of n is even so that $n(n^2+1)$ is even.

If n is odd, then n^2 is odd and n^2+1 is even so that $n(n^2+1)$ is again even.

5. Devise a rule for divisibility by 25.

6. The number 4321 is not divisible by 11. How many numbers divisible by 11 can you construct by permuting the four digits. For example. 2431 is divisible by 11. There will be eight such numbers. Find the others.

7. If a and b are two numbers having odd number of digits such that b has the same digits in opposite order of a, then show that $(a-b)$ is divisible by 11.

8. If a and b in problem 7 above have even number of digits, then show that $a+b$ is divisible by 11.

(IV) Divisibility by 7, 11, 13.

Since $7 \times 11 \times 13 = 1001$, we can develop an interesting rule for determining the divisibility of a number by 7, 11 or 13.

Consider a number 654,489.

$$\begin{aligned} 654{,}489 &= (654)10^3 + 489 \\ &= (654)(1001-1)+489 \\ &= 654 \times 1001 - 654 + 489 \\ &= 653 \times 1001 - 165 \\ &\cong -165 \pmod{1001} \end{aligned}$$

In other words, on dividing 654,489 by 1001, it leaves a remainder of -165 (which means that the number is short by 165 for divisibility by 1001).

Since 7, 11, 13 are divisors of 1001, therefore any multiple of 1001 would be divisible by 7, 11, 13. If the remainder (in this case -165) is divisible by 7 and/or 11 and/or 13, then the number is also divisible.

Take the general form of a number *i.e.*,

$$a_n.10^n + \ldots\ldots + a_3.10^3 + a_2.10^2 + a_1.10 + a_0$$

and develop the following rule :

> To test the divisibility of a number by 7, 11 or 13, divide the number in blocks of three digits beginning from the unit's digit. Add the numbers of the alternate blocks. If the difference of two sums is divisible by 7 or 11 or 13, then the number is also accordingly divisible by 7 or 11 or 13.

Example (*i*) *57,713,483,984,560,138 is divisible by 7, but not by 11 or 13.*

57	713	
483	984	because $1835-1100=735$ is
560	138	divisible by 7 but not by 11
——	——	or 13.
1100	1835	

(ii) 512,215,312,213,105,578 is divisible by 7 and 11 but not by 13.

512	215		
312	213	1006	because $1006-929=77$ is divisible by 7 and 11 but not by 13.
105	578	-929	
929	1006	77	

Problem Set 2.3

1. Show that the following numbers are divisible by 1001 :
358,358 ; 506,506 ; 525,525 ; 4,368,364 ; 5,492,487.

2. Show that the following numbers are divisible by 7 :
358,365 ; 506,569 ; 25,795 ; 6,506,010 ; 56,537,404.

3. Which of the numbers in problem 2 above are divisible by 11 also ?

4. Which of the following numbers are divisible by 13 :
130,143 ; 143,286 ; 506,518 ; 130,103 ; 250,276.

5. If a number is not divisible by 7, then neither of its nth multiple, $n=1, 2, 3, 4, 5, 6$ is divisible by 7.

First consider numbers 71, 72, 73, 74, 75, 76. Then try to build a logical proof.

6. Show that if a number is not divisible by 11, then neither of its nth multiple, $n=1, 2, 3, \ldots 10$ is divisible by 11.

7. Show that a number is not divisible by 13, then neither of its nth multiple, $n=1, 2, \ldots 12$ is divisible by 13.

(*V*) *Some Miscellaneous Rules*

"Two numbers are called "prime to each other" or "coprimes" if they have no common factor except 1."

Example. 5, 6 ; 3, 4 ; 11, 18 ; 15, 77, etc.

We can show that "If a number is divisible by two coprimes m and n, then it is also divisible by their product."

Example. 36 is divisible by 2 and also by 3, so it is also divisible by 2×3.

Proof. Let a number p be divisible by m and n such that $(m, n)=1$

[(m, n) is read as H.C.F. of m and n]

$$p=q_1.m$$

Since p is also divisible by n, at least one of its factors out of q_1 and m must be divisible by n.

Since n does not divide m, it must divide q_1.

Let $\qquad q_1 = q_2.n$

so that $\qquad p = q_1.m = q_2.nm$

Hence p is divisible by mn also.

Problem Set 2.4

1. Find out rules for divisibility by 6, 12, 15, 30, 75.

2. Find what numbers less than 7 divide the following :

913,312 ; 4,312 ; 612,513 ; 373,835.

3. Show that the sum and product of any three consecutive integers is divisible by 3.

4. Show that $n(n+1)(n+2)$ is divisible by 6 where n is a natural number.

5. Show that the product of any four consecutive integers is divisible by (*i*) 4, (*ii*) 8, (*iii*) 12 and (*iv*) 24.

6. Show that the product of n consecutive integers, $n \geqslant 4$ is always divisible by (*i*) n, (*ii*) $2n$, (*iii*) $3n$, and (*iv*) $6n$.

3

Partition of Numbers

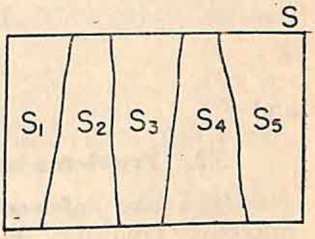
3.1. Partition of Sets

If S_1, S_2, . S_n are any sub-sets of set S such that :

(*i*) There are no common elements in any of the subsets ;

(*ii*) The union of all the subsets gives the set S :

then we say that S_1, S_2,...S_n are partitions of S.



Examples. (*i*) If N_e is the set of even natural numbers and N_o is the set of odd natural numbers, then we have :

(*a*) N_e and N_o have no common element, and

(*b*) $N_e \cup N_o = N$

so that N_e and N_o are partitions of N.

It may be interesting to note that the above partition results from the relation "Leaves the same remainder on division by 2". In N_e, we have all natural numbers which leave a remainder of "zero" on division by 2 and in N_o, we have all natural numbers which leave a remainder of "1" on division by 2.

(*ii*) Find the partitions of N as a result of the relation "leaves the same remainder on division by 3".

We have $N_0 = \{ x : x = 3n, n \in N \}$;

$N_1 = \{ x : x = 3n+2, n \in N \}$;

$N_2 = \{ x : x = 3n+2, n \in N \}$.

It is obvious that

$$N_0 \cap N_1=\phi,$$

$$N_1 \cap N_2=\phi$$

and

$$N_0 \cap N_2=\phi$$

and

$$N_0 \cup N_1 \cup N_2=N$$

We have

$$N_0=\{ 3, 6, 9,... \};$$

$$N_1=\{ 1, 4, 7... \};$$

$$N_2=\{ 2, 5, 8, 11,... \}$$

(*iii*) The idea of partition in example (*i*) and (*ii*) above could be extended to the set of integers also, so that in example (*i*) above

$$I_e=\{ 0, 2, -2, 4, -4,... \}$$

and

$$I_o=\{ 1, -1, 3, -3, 5, -5, .. \}$$

In example (*ii*) above

$$I_0=\{ 0, 3, -3, 6, -6,... \},$$

$$I_1=\{ 1, -2, 4, -5, 7, -8, .. \}$$

and

$$I_2=\{ 2, -1, 5, -4, 8, -7, .. \}$$

3.2. Problems involving Partitions

The idea of partition and divisibility together give rise to interesting problems. We will discuss some of them here.

Problem 1. *If the square of an integer is not divisible by 3, then it must leave a remainder of 1.*

Solution.

Partition the set of integers by the relation "leaves the same remainder on division by 3". We have three subsets namely

$$I_0=\{ 0, 3, -3, 6, -6,... \}$$

$$I_1=\{ 1, -2, 4, -5, 7, -8,... \}$$

and

$$I_2=\{ 2, -1, 5, -4, 8, -7,... \}$$

Now it is obvious that any member of I_0 is congruent to 0 mod 3

e.g.,
$$-15=-5\times3+0$$
$$\cong 0 \pmod 3$$

Similarly any member of I_1 is congruent to 1 (mod 3)

and any member of I_2 is congruent to 2 (mod 3).

If we denote members of these three subsets as i_0, i_1 and i_2.

Then $\quad i_0 \cong 0 \pmod 3$;

$i_1 \cong 1 \pmod 3$

and $\quad i_2 \cong 2 \pmod 3$

So that $\quad i_0^2 \cong i_0 \times 0 \pmod 3$

$\cong 0 \times 0 \pmod 3$;

$i_1 \cong 1 \pmod 3$

$i_1^2 \cong i_1 \times 1 \pmod 3$

$\cong 1 \times 1 \pmod 3$;

$i_2 \cong 2 \pmod 3$

$i_2^2 \cong i_2 \times 2 \pmod 3$

$\cong 2 \times 2 \pmod 3$

$\cong 1 \pmod 3$ $\qquad$ because $4 \cong 1 \pmod 3$

If we explain the above result in words, we may say that on dividing by 3, any members of I_0 class, when squared leaves a remainder "0" and any member of I_1 or I_2 class when squared leaves a remainder "1".

In other words,

If the square of an integer is not divisible by 3, then it must leave a remainder of 1.

Problem 2. *If the cube of a number is not divisible by 9, then it must leave a remainder of 1 or 8.*

Solution.

Partition the set of integers by the relation "leaves the same remainder on division by 9". On the pattern of the preceding example, we have nine classes namely

$$I_0 = \{0, 9, -9, 18, -18, \ldots\}$$
$$I_1 = \{1, 10, -8, 19, -17, \ldots\}$$

and so on.

Now if i_0, i_1, i_2,...be any number belonging to classes I_0, I_1, I_2,...

Then $i_0 \cong 0 \pmod 9$ so that $i_0^3 \cong 0 \pmod 9$

$i_1 \cong 1 \pmod 9$ so that $i_1^3 \cong 1 \pmod 9$

$i_2 \cong 2 \pmod 9$ so that $i_2^3 \cong 2 \times 2 \times 2 \pmod 9 \cong 8 \pmod 9$

$i_3 \cong 3 \pmod 9$ so that $i_3^3 \cong 3 \times 3 \times 3 \pmod 9 \cong 0 \pmod 9$

$i_4 = 4 \pmod 9$ so that $i_4^3 = 4 \times 4 \times 4 \pmod 9 \cong 1 \pmod 9$

$i_5 = 5 \pmod 9$ so that $i_5^3 = 5 \times 5 \times 5 \pmod 9 \cong 8 \pmod 9$

$i_6 = 6 \pmod 9$ so that $i_6^3 = 6 \times 6 \times 6 \pmod 9 \cong 0 \pmod 9$

$i_7 \cong 7 \pmod 9$ so that $i_7^3 = 7 \times 7 \times 7 \pmod 9 \cong 1 \pmod 9$

$i_8 = 8 \pmod 9$ so that $i_8^3 = 8 \times 8 \times 8 \pmod 9 \cong 8 \pmod 9$

Summarising the above result, we can say that

$$\left.\begin{array}{llll} i_0^3,\ i_3^3 \text{ and } i_6^3 \text{ are each} & \text{congruent} & \text{to} & \text{zero} \pmod 9 \\ i_1^3,\ i_4^3 \text{ and } i_7^3 \text{ are each} & ,, & ,, & \text{one} \pmod 9 \\ i_2^3,\ i_5^3 \text{ and } i_5^3 \text{ are each} & ,, & ,, & \text{eight} \pmod 9 \end{array}\right\} \quad \ldots(A)$$

Hence, if the cube of an integer is not divisible by 9, then it must leave a remainder of 1 or 8.

Alternate Method. Result at (A) suggests that we should be able to prove the above result by partitioning numbers with respect to divisibility by 3 also.

The entire set of integers can be partitioned into three categories, namely : $3n$; $3n+1$; $3n+2$ such that n is an integer.

So that $(3n)^3 = 27n^3 \cong 0 \pmod 9$

$$\begin{aligned} (3n+1)^3 &= 27n^3 + 27n^2 + 9n + 1 \\ &= 9\,(3n^3 + 3n^2 + n) + 1 \\ &\cong 1 \pmod 9 \\ (3n+2)^3 &= 27n^3 + 54n^2 + 36n + 8 \\ &= 9(3n^3 + 6n^2 + 4n) + 8 \\ &\cong 8 \pmod 9 \end{aligned}$$

Hence the result.

Problem Set 3.1

1. Show that the square of any integral power of an even integer is even and of an odd integer is odd.

[**Hint**. Partition the set of integers into two subsets with respect to divisibility by 2, and prove.]

2. Show that the square of no integer ends in 2 or 8.

Solution. Partition the set of integers with respect to divisibility by 10. We have ten partitions. Of these I_0, I_2, I_4, I_6, I_8 contain even numbers whereas I_1, I_3, I_5, I_7, I_9 contain odd numbers.

Since the squares of even numbers only are even, we shall consider only the first set of partitions.

Now if i_0, i_2, i_4, i_6, i_8 are numbers belonging to I_0, I_2, I_4, I_6 and I_8 respectively, then

$i_0 \cong 0 \pmod{10}$ hence $i_0^2 \cong 0 \pmod{10}$

$i_2 \cong 2 \pmod{10}$ hence $i_2^2 \cong 4 \pmod{10}$

$i_4 \cong 4 \pmod{10}$ hence $i_4^2 \cong 4 \times 4 \pmod{10} \cong 6 \pmod{10}$

$i_6 \cong 6 \pmod{10}$ hence $i_6^2 \cong 6 \times 6 \pmod{10} \cong 6 \pmod{10}$

and $i_8 \cong 8 \pmod{10}$ hence $i_8^2 \cong 8 \times 8 \pmod{10} \cong 4 \pmod{10}$.

Summarising the above result we may say that square of any even integer must end in 0, 4 or 6.

Since the square of any odd integer is always odd, hence the square of no integer ends in 2 or 8.

3. Find by inspection which of the following number s cannot be squares of whole numbers :

349252, 405642, 31366, 545454, 386128.

4. Show that if a number n is not divisible by 5, then n^2+1 or n^2-1 must be divisible by 5.

[**Hint**. Partition the set of integers with respect to divisibility by 5. Then

$i_1^2 \cong 1 \pmod 5$; $i_4^2 \cong 4 \times 4 \pmod 5 \cong 1 \pmod 5$;

$i_2^2 \cong 2 \times 2 \pmod 5 \cong -1 \pmod 6$;

$i_3^2 \cong 3 \times 3 \pmod 5 \cong -1 \pmod 5$

i.e., any number belonging to I_1 or I_4 leaves a remainder of 1 when divided by 5 and any number belonging to I_2 or I_3 is short by 1 for divisibility by 5.

Hence you get the above result.]

5. If a number n is not divisible by 7, then either n^3+1 or n^3-1 is divisible by 7.

6. If a number n is not divisible by 11, then either n^5+1 or n^5-1 is divisible by 11.

7. Show that the only even numbers that can possibly be perfect fourth powers must end in zero or 6.

8. Show that the only odd numbers that can possibly be perfect fourth powers must end in 1 or 5.

9. Show that the only even numbers that can possibly be perfect $4n^{th}$ powers must end in zero or 6, where $n \in N$.

10. Show that the only odd numbers that can possibly be perfect $4n^{th}$ powers must end in 1 or 5, where $n \in N$.

4

Prime Numbers

4.1. Prime Numbers.

Prime numbers are those natural numbers which have no other factors except 1 and itself.

Examples. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, etc.

Note—1 is not considered as a prime. It has exactly one factor *i.e.* 1.

Problem 1. *Show that 2 is the only even nnmber which is a prime.*

Solution.

All even numbers greater than 2 have at least 1, 2 and the number itself as factors. Hence they are not prime.

Twin primes.

If two successive odd numbers are prime, they are known as **twin** primes.

Examples. (3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43), (59, 61) and so on.

Prime triplets.

If three successive odd numbers are prime, they are known as prime triplets.

Example. (3, 5, 7)

Problem 2. *Show that (3, 5, 7) is the only prime triplet.*

Solution.

(3, 5, 7) is a prime triplet ...by definition.

Now consider three successive odd numbers

$$x,\ x+2 \text{ and } x+4, \quad x>3.$$

Either $3 \mid x$ (read as 3 divides x) or $3 \nmid x$ (read as 3 does not divide x)

If $3 \mid x$, then $(x, x+2, x+4)$ do not form a prime triplet.

If $3 \nmid x$, then $x \cong 1 \pmod 3$ or $x \cong 2 \pmod 3$.

If $x \cong 1 \pmod 3$, then $x+2 \cong 1+2 \pmod 3 \cong 0 \pmod 3$ so that $(x, x+2, x+4)$ do not form a prime triplet.

If $x \cong 2 \pmod 3$, then $x+4 \cong 2+4 \pmod 3 \cong 0 \pmod 3$; hence again $(x, x+2, x+4)$ do not form a prime triplet.

Hence no three consecutive odd numbers $x, x+2, x+4, x>3$ form a prime triplet.

Locating Prime Numbers

Problem 3. *Show that any prime number greater than 3 belongs to one of the following classes—*

(*i*) *$6n-1$, $n \in N$ i.e. if 1 is added to the number the sum is divisible by 6.*

(*ii*) *$6n+1$, $n \in N$ i.e. if 1 is subtracted from the number, the difference is divisible by 6.*

Solution.

Partition the set of natural numbers with respect to divisibility by 6. Then the six partitions so formed will contain numbers represented by :

$$6n \; ; \; 6n+1 \; ; \; 6n+2 \; ; \; 6n+3 \; ; \; 6n+4 \; ; \; 6n+5$$

It is obvious that the numbers represented by $6n$, $6n+2$, $6n+3$ and $6n+4$ must be composite.

Numbers represented by $6n+1$ and $6n+5$ may or may not be composite.

Hence the only classes of numbers that contain prime numbers are represented by $6n+1$ and $6n+5$ ($6n+5$ may be expressed equivalently as $6n-1$ without loss of generality.)

Now $6n+1 \cong 1 \pmod 6$

and $6n+5 \cong 5 \pmod 6 \cong -1 \pmod 6$

Hence the result.

4.2. To find out whether any given number is Prime or Composite

The obvious way of finding out whether a number is prime or composite is to find out whether it is divisible by some number other than 1 and itself.

Suppose we want to test 899 for primality or otherwise.

The number is not divisible by 2.

Will it be divisible by any multiple of 2 ? No.

Hence we need not consider the set of divisors

$$\{ 4, 6, 8, 10, 12, \ldots \}$$

The number is not divisible by 3, hence it will not be divisible by any of its multiples. Hence we need not consider the set of divisors

$$\{ 6, 9, 12, 15, 18, \ldots \}$$

This gives us the first rule :

> For testing the primality of any given number, check it for divisibility by prime numbers 2, 3, 5, 7, ...

Our next problem is that if the given number is a prime, we will not get any divisor except 1 or the number itself. Shall we continue checking for divisibility till we come to the number itself ? To answer this question consider the following :

Example. Find the set of divisors of 72.

We have

$$\begin{aligned}
72 &= 2 \times 36 \\
72 &= 3 \times 24 \\
72 &= 4 \times 18 \\
72 &= 6 \times 12 \\
72 &= 8 \times 9 \\
72 &= 9 \times 8 \\
72 &= 12 \times 6 \\
72 &= 18 \times 4 \\
72 &= 24 \times 3 \\
72 &= 36 \times 2
\end{aligned}$$

If you can see the pattern of factors, you will notice that after the pair of divisors (8, 9), next is (9, 8) and so on. This means that as we go further, the same factors appear again.

Observe again in the following :

$384 = 2 \times 192$	or	192×2
$384 = 3 \times 128$	or	128×3
$384 = 4 \times 96$	or	96×4
$384 = 6 \times 64$	or	64×6
$384 = 8 \times 48$	or	48×8
$384 = 12 \times 32$	or	32×12
$384 = 16 \times 24$	or	24×16

From the above two examples, it is clear that we need not go on dividing by primes till we reach the number itself. We can stop somewhere in between. But where ?

We have $16 \times 24 = 384$

so that $16 \times 16 < 384$

and $24 \times 24 > 384$

i.e., 16 is less than the square root of 384

and 24 is greater than the square root of 384.

This gives us the rule :

> For testing the primality of any given number N, check it for divisibility by prime numbers 2, 3, 5, 7, ... till we arrive at a prime number next lower than $\sqrt{N}$.

Problem 4. *Check if 10139 is a prime number ?*

Solution.

We have $\sqrt{10139} = 100.$ ......

so that we need to check by prime number less than 100.

It is not divisible by 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 91, 97.

Hence it is a prime number.

However, we are required to carry out a number of cumbersome divisions. Let us, therefore, consider some short-cuts which may be useful in some cases.

Let N be the given number and let m be the least number for which $m^2 > N$. From the numbers :

$$m^2 - N \,;\, (m+1)^2 - N \,;\, (m+2)^2 - N, \ldots$$

When we arrive at a number which is a perfect square,

we get $x^2 - N = y^2$

so that $N = x^2 - y^2$

$= (x-y)(x+y)$.

Problem 5. *Find out if 9271 is a prime number.*

Solution.

We have $96 < \sqrt{9271} < 97$.

So that we shall start with 97

$$97^2 - 9271 = 138$$
$$98^2 - 9271 = 333$$
$$99^2 - 9271 = 530$$
$$100^2 - 9271 = 729 = 27^2$$

so that $9271 = 100^2 - 27^2$

$= (100-27)(100+27)$

$= 73 \times 127$.

This method is particularly useful if the number N has a factorization in which the two factors are of about the same magnitude, since then the difference is small.

If N is itself a prime, the only possible factors could be N and 1. In that case $N = (x+y)(x-y)$ is possible only if $x+y = N$ and $x-y = 1$.

4.3. Some more information on Prime Numbers.

Theorem. *To show that the set of prime numbers is infinite.*

Proof. Suppose that we have only a finite number of primes.

Let P be the largest prime number when this finite set of primes is arranged in order of increasing magnitude so that this set $M = \{2, 3, 5, 7, 11, \ldots P\}$.

Let $N = (2 \times 3 \times 5 \times 7 \times 11 \times \ldots \times P) + 1$

N exceeds any prime number in the given set M, so that it is not found in M, hence N should be composite.

However N is not divisible by any of the primes in set M (Why ?) and so it should be divisible by a prime not in M.

Hence M does not contain every prime.

Therefore we prove by contradiction, that the set of primes is not finite.

Prime Decades with Four or More Primes

A decade is a set of ten continuous numbers. Numbers from 1 to 10 form the first decade ; 11 to 20 form the second decade and so on.

What is the maximum number of primes in a decade ? Except for 2, all other even numbers are divisible by 2 and hence cannot be prime. The only primes can, therefore, exist in the odd natural numbers. Of these, the number ending in 5 is divisible by 5 and hence cannot be prime.

The remaining numbers ending in 1, 3, 7, 9 are often divisible by 3 or 7 or 9, *e.g.* 33, 63, 93 are divisible by 3 ; 21, 51, 81, 111 are also divisible by 3 ; 49, 119 .. are divisible by 7, etc. Hence decades with four primes are very rare. Some of these are :

Decade	*Primes*
1 to 10	2, 3, 5, 7
11 to 20	11, 13, 17, 19
101 to 110	101, 103, 107, 109
191 to 200	191, 193, 197, 199
821 to 830	821, 823, 827, 829
1481 to 1490	1481, 1483, 1487, 1489
1871 to 1880	1871, 1873, 1877, 1879
⋮	⋮

Goldbach's Conjecture. In 1742, Christian Goldbach in a letter to Euler suggested that every even number from 6 onwards is representable as the sum of two prime numbers other than 2.

$6=3+3$; $8=3+5$; $10=3+7$; $12=5+7$; $14=7+7$

$16=5+11$; $18=5+13$; $20=7+13$; $22=5+17$; $24=7+17$

$26=3+23$; $28=5+23$; $30=7+23$; $32=3+29$; $34=5+29$

$36=7+29$; $38=7+31$; $40=11+29$; $42=11+31$; $44=13+31$

and so on.

Although about two hundred and thirty three years have passed, nobody has been able to prove Goldbach's conjecture nor disprove it. Difficulty lies in the fact that the properties of primes are defined in terms of multiplication and as such any problem like this which relates to additive properties of primes is necessarily difficult.

However sufficient progress has been made recently and some day we may find the conjecture proved or disproved.

Problem Set 4.1

1. Show that the sum of twin primes greater than 3 is always divisible by 12.

2. Find out if the following numbers are prime :

947, 883, 3043, 8111

3. We have

$2 \qquad\qquad\quad +1=3$ a prime number

$2\times 3 \qquad\quad +1=7$,, ,,

$2\times 3\times 5 \quad +1=31$,, ,,

$2\times 3\times 5\times 7+1=211$,, ,,

Check up if the following are prime numbers ?

$2\times 3\times 5\times 7\times 11+1 = \ldots$

$2\times 3\times 5\times 7\times 11\times 13+1=\ldots$

$2\times 3\times 5\times 7\times 11\times 13\times 17+1=\ldots$

4. Express even numbers from 40 to 100 as the sum of two primes other than 2.

5. Show that if a prime number greater than 2 is squared and divided by 4, it always leaves a remainder of 1.

6. Show that if a prime number greater than 2 is squared and divided by 8, it always leaves a remainder of 1.

7. Show that if a prime number greater than 3 is squared and divided by 3, it always leaves a remainder of 1.

8. Show that if a prime number greater than 2 is squared and divided by 6, it always leaves a remainder of 1.

9. Show that if a prime number greater than 3 is squared and divided by 12, it always leaves a remainder of 1.

5

Arithmetic of Different Bases

5.1. Positional Notation of Numbers.

The numeration system that is internationally used today is a gift to the humanity from Hindu mathematicians. It is they who gave to the world the notion of '0' (called zero) and a numeration system using only ten symbols from 0 to 9 and a place-value concept that made it possible to express any number with the help of these ten symbols only. Since this numeration system only uses ten symbols, we call it decimal numeration system.

Consider any number say 359. Then in expanded notation, this would be written as :

$$3\times 10^2+5\times 10+9$$

Decimal numeration system is a specific example of the use of positional notation. If we were to use fewer or more symbols than ten, to express a number, the same number would be expressed differently, *e.g.*, in base two, we use only two symbols namely 0 and 1, so that 13 will be expressed as 1101_{two} because

$$\begin{aligned} 1101_{two} &= 1\times 2^3+1\times 2^2+0\times 2+1 \\ &= 8+4+1 \\ &= 13. \end{aligned}$$

Conversion.

Conversion from base ten to any other base can be done by successive division as below :

Problem 1. *Convert 143 into base two.*

Solution.

$$
\begin{array}{r|rl}
 & & \text{remainder} \\
2 & 143 & ,\ 1 \\
2 & 71 & ,\ 1 \\
2 & 35 & ,\ 1 \\
2 & 17 & ,\ 1 \\
2 & 8 & ,\ 0 \\
2 & 4 & ,\ 0 \\
2 & 2 & ,\ 0 \\
2 & 1 & ,\ 1 \\
 & 0 &
\end{array}
$$

We have $143 = 10001111_{two}$

Check

$$10001111_{two} = 1\times2^7+0\times2^6+0\times2^5+0\times2^4+1\times2^3+1\times2^2+1\times2+1$$
$$=128+8+4+2+1$$

Problem 2. *Convert 935 into base five.*

Solution.

$$
\begin{array}{r|rl}
 & & \text{remainder} \\
5 & 935 & ,\ 0 \\
5 & 187 & ,\ 2 \\
5 & 37 & ,\ 2 \\
5 & 7 & ,\ 2 \\
5 & 1 & ,\ 1 \\
 & 0 &
\end{array}
$$

Hence $935 = 12220_{five}$

Check

$$12220_{five} = 1\times5^4+2\times5^3+2\times5^2+2\times5+0$$
$$=625+250+50+10$$
$$=935$$

Here are some numbers written in different bases.

Number \ Base	*ten*	*two*	*three*	*four*	*five*	*seven*	*nine*	*twelve*
one	1	1	1	1	1	1	1	1
two	2	10	2	2	2	2	2	2
three	3	11	10	3	3	3	3	3
fou	4	100	11	10	4	4	4	4
five	5	101	12	11	10	5	5	5
six	6	110	20	12	11	6	6	6
seven	7	111	21	13	12	10	7	7
eight	8	1000	22	20	13	11	8	8
nine	9	1001	100	21	14	12	10	9
ten	10	1010	101	22	20	13	11	*t*
eleven	11	1011	102	23	21	14	12	*e*
twelve	12	1100	110	30	22	15	13	10

Note. (*i*) For writing in base twelve, we need two more symbols for which *t*, *e* have been used arbitrarily.

(*ii*) In order to distinguish between numerals in different bases, we read them differently from numerals in base ten. For example, 101_{two} will be read as "one, zero, one base two" and so on.

It appears that in the past, some efforts have been made to count in bases other than five. For example in India, there is a widely prevalent practice of counting objects in five's. Dozen and gross remind us of base twelve.

Addition, subtraction, multiplication and division in other bases follow exactly the same pattern as in base ten, although the

processes appear to be difficult for the simple reason that we are used to computations in base ten only.

Consider the following examples.

Example.

Base two	*Base three*	*Base five*
1011011	1011011	1011011
+ 110101	+ 110101	+ 110101
10010000	1121112	1121112

Please note that although we have used the same numerals in the three addition examples above, they actually denote different numbers.

We shall now take any two numbers, express them in different bases, add them and then check the results.

Example

Base ten	*Base five*	*Base eight*	*Base twelve*
349	2344	535	251
+164	+1124	+244	+118
513	4023	1001	369

Check. $4023_{five}{}^{*} = 4\times5^3+2\times5+3 \quad = 500+10+3 = 513$

$1001_{eight} = 1\times8^3+1 \quad = 512+1 \quad = 513$

$369_{twelve} = 3\times12^2+6\times12+9 = 432+72+9 = 513$

Problem 3. *Write the multiplication table of 8 in base nine.*

base nine

$8\times1 = 8$

$8\times2 = 17$

$8\times3 = 26$

$8\times4 = 35$

$8\times5 = 44$

$8\times6 = 53$

$8\times7 = 62$

$8\times8 = 71$

$8\times9 = 80$

*Whenever we express a number in a base other than ten, we suffix the base (in letters) to the numeral denoting the number.

When there is no such indication, the base is to be considered as ten.

Problem 4. *Carry out multiplication in base nine.*

base nine

$$\begin{array}{r} 1234568 \\ \times 8 \\ \hline 11111111 \end{array}$$

We may explain it as below :

$8 \times 8 = 71$

$6 \times 8 = 53$, $53 + 7 = 61$

$5 \times 8 = 44$, $44 + 5 = 51$

$4 \times 8 = 35$, $35 + 4 = 41$

$3 \times 8 = 26$, $26 + 3 = 31$

$2 \times 8 = 17$, $17 + 2 = 21$

$1 \times 8 = 8$, $8 + 2 = 11$

Note that the above product is very similar to the one that we get when we multiply 12345679 by 9 in base ten. Can you discover some pattern and construct similar examples in other bases.

5.2. Divisibility Rules in Other Bases.

In the chapter of divisibility rules, we have shown (using congruence of numbers) that a number (written in base ten) is divisible by nine if and only if the sum of the digits of that number is divisible by nine. The same rule applies for 3, because 3 is a factor of 9.

It is interesting to note what the divisibility rules will look like, if we were to express our numbers in some other base. We first of all consider base nine.

We have

$$40_{nine} = 4 \times 9 + 0 = 36$$

$$50_{nine} = 5 \times 9 + 0 = 45$$

$$8120_{nine} = 8 \times 9^3 + 1 \times 9^2 + 2 \times 9 + 0$$

$$= 5931$$

It is obvious that the above numbers are divisible by 9.

We have therefore, the rule :

"A number written in base nine is divisible by nine iff the unit's digit is zero"

You can see a pattern, if you consider the following :

A number written in base ten is divisible by ten iff the unit's digit is zero.

These rules are really specific cases of a more general rule :

A number written in base 'n' is divisible by 'n' if and only if the unit's digit is zero.

Example. 30_{five}, 440_{five} are divisible by five.

4230_{seven}, 360_{seven} are divisible by seven and so on.

Let us consider other divisibility rules for numbers written in base nine.

Consider a number $a\ b\ c_{(nine)}$ where a, b, c are digits of a given number. The extended notation for the number is :

$$a \times 9^2 + b \times 9 + c$$

Now $\quad a \times 9^2 \cong a \pmod{8}$

$\quad b \times 9 \cong b \pmod{8}$

and $\quad c \cong c \pmod{8}$

hence $\quad a \times 9^2 + b \times 9 + c = a + b + c \pmod{8}$

We therefore have the rule :

A number written in base nine is divisible by 8, if and only if the sum of its digits is divisible by 8.

Since 2 and 4 are factors of 8, we will have the same rule applicable for 2 and 4 also, *i.e.*

A number written in base nine is divisible by a factor of 8 (including itself) iff the sum of its digits is divisible by that factor.

The above two rules are specific cases of a general rule :

A number written in base 'n' is divisible by $(n-1)$ and its factors, iff the sum of its digits is divisible by that factor.

Example. 3427_{nine} is divisible by 2, 4 and 8, because the sum of its digits is 17_{nine} *i.e.* 16 (in base ten)

and $\quad 16 \cong 0$ (mod 8 or mod 4 or mod 2)

[**Check**—$3427_{nine} = 3\times 9^3 + 4\times 9^2 + 2\times 9 + 7$

$$= 2187 + 324 + 18 + 7$$

$$= 2536$$

In base ten, the number will be expressed as 2536. You can now check the statement for yourself.]

Problem 5. *Show that 3351_{seven} is divisible by 2, 3 or 6.*

Solution.

Let $n=7$,

then $n-1=6$

So that 3351_{seven} will be divisible by 2, 3 or 6 if and only if the sum of its digits is divisible by these numbers.

Since $3+3+5+1=12$ which is divisible by 2, 3 or 6, hence 3351_{seven} is divisible by 2, 3 or 6.

[**Note**. Convert the number into base ten and check the above statement.]

From the above examples it should be clear that in bases other than ten, a number ending in 2, 4, 6, 8 or 0 is not necessarily an even number and a number ending in 1, 3, 5, 7, 9 is not necessarily an odd number.

5.3. Some Miscellaneous Results.

Problem 6. *Show that in any base greater than 3, 231 is divisible by 11 and by 21 written in that base.*

Solution.

Let the base be a

Then $231 = 2a^2 + 3a + 1$

$$= (a+1)(2a+1)$$

Now $a+1 = 1.a+1$ which is the expanded notation for $11_{base\ a}$

and $2a+1$ is the expanded notation for $21_{base\ a}$.

Hence $231_{base\ a}$ is divisible by $11_{base\ a}$ and $21_{base\ a}$.

An interesting game—

There is an interesting game of choosing a number from a given list, which can be found out.

Make a table of numbers as given below :

I	*II*	*III*	*IV*	*V*
1	2	4	8	16
3	3	5	9	17
5	6	6	10	18
7	7	7	11	19
9	10	12	12	20
11	11	13	13	21
13	14	14	14	22
15	15	15	15	23
17	18	20	24	24
19	19	21	25	25
21	22	22	26	26
23	23	23	27	27
25	26	28	28	28
27	27	29	29	29
29	30	30	30	30
31	31	31	31	31

Ask a person to select a number of his choice from amongst the given table and to keep it unto himself. Ask him to name the columns in which it occurs. Suppose he chooses 18. Then 18 appears in II and V columns only. Add numbers of the first row in column II and V *i.e.* $2+16$ which gives you 18.

Let us take another number say 30. It appears in columns II, III, IV and V. So adding $2+4+8+16$, we get 30.

Explanation. In the first example above, we took 18.

Now 18 in base two will be written as 10010_{two} *i.e.* we have *a* "one" only in the second and fifth column beginning from the left.

$$10010=1\times2+1\times2$$
$$=1\times16+1\times2$$
$$=16+2$$

Note that 16 and 2 are the place-values in the II and V column in base two. Since the first row in each column in the given table contains the numbers indicating place-value of each column hence adding the first row numbers of II and V column gives the desired number.

We may verify this for 30 also.

$$30=11110_{two}$$
$$11110_{two}=1\times2^4+1\times2^3+1\times2^2+1\times2$$
$$=16+8+4+2$$

Adding the first row numbers of II, III, IV and V columns in which 30 occurs in the table, we get $16+8+4+2=30$.

Another game based on base three

You can construct a similar game based on base three.

We have $$26=222_{three}$$
$$=2\times3^2+2\times3+2$$
$$=2\times9+2\times3+2$$
$$=(9+9)+(3+3)+(1+1)$$

If we were to restrict our numbers from 1 to 26, then we need only three columns. Place value of I, II and III columns will be 1, 3 and 9 respectively. However, a number like 26 has to be repeated twice in all the three columns, whereas a number like 13 will appear only once in each column. The following table makes two sub-columns in each column. Examine how different numbers have been chosen in each column and sub-column.

Ia	Ib	IIa	IIb	IIIa	IIIb
1	1	3	3	9	9
1	2	3	6	9	18
2	5	4	7	10	19
4	8	5	8	11	20
5	11	6	15	12	21
7	14	7	16	13	22
8	17	8	17	14	23
10	20	12	24	15	24
11	23	13	25	16	25
13	26	14	26	17	26
14		15		18	
16		16		19	
17		17		20	
19		21		21	
20		22		22	
22		23		23	
23		24		24	
25		25		25	
26		26		26	

Consider a number say 21.

It appears in columns IIa, IIIa and IIIb, so that $3+9+9=21$.

Take another number say 17.

It occurs in columns Ia, Ib, IIa, IIb and IIIa.

Hence $1+1+3+3+9=17$

Similarly you may try any other number between 1 to 26.

Similar tables can be constructed for higher bases. However the number of sub-columns increase as we go up, and as such the tables may begin to appear unwieldy.

Problem Set 5·1

1. Write numbers from one to sixteen in base two, base three and base five.
2. Write numbers from one to fifteen in base two and four.
3. Perform the following additions :

Base two	Base three	Base five	Base seven
1101000	21101	32014	56123
+101111	+11011	+41032	+1026

4. Develop multiplication tables for five in base eight, seven, six, five and four, and check their correctness.
5. Perform the following multiplication :

Base five	Base six	Base seven	Base eight
124	1235	12346	123457
×4	×5	×6	×7
124	1235	12346	123457
×13	×14	×15	×16

6. Can you see some pattern in the examples given in problem 5 above. Find similar examples in base nine and ten.
7. Does an odd number necessarily end in an odd digit in all numeration systems with different bases ? Give examples.
8. Does an even number necessarily end in an even digit in all numeration systems with different bases ? Give examples.
9. Under what condition will a number written in base "a" be divisible by

   (*i*) a

   (*ii*) $a-1$

   (*iii*) $a+1$

10. State the rule for divisibility by 2, 4 and 8 for numbers written in base nine.

11. State the rule for divisibility by 2 and 3 for numbers written in base seven.

12. State the rule for divisibility by 2 and 4 for numbers written in base five.

13. Check if the following statements are true :

(*i*) 156 is divisible by 13 written in any base greater than six.

(*ii*) 672 is divisible by 21 and 32 written in any base greater than seven.

(*iii*) 10,101 is divisible by 111 written in any base.

14. Show that 736 is divisible by 23 and 32 if the base is ten. However this is not true in any other base. Give reasons.

15. Show that 1573 is divisible by 121 if the base used is greater than eight.

16. Show that

(*i*) 144 is a perfect square in all bases greater than four.

(*ii*) 169 is a perfect square in all bases greater than nine.

(*iii*) 441 is a perfect square in all bases greater than four.

(*iv*) 1331 is a perfect cube in any base greater than three.

17. The game "Select a number" discussed in this chapter is based on base two. You can slightly vary the format by selecting pink, blue and green cards for the three columns. Write numbers on these cards and then play the game.

18. Another variation of the game can be effected by using metal sheets of 1 unit weight, two units weight, four units weight, eight units weight and sixteen units weight respectively. We may retain the same size of the metal sheets and only vary the thickness. Write the correct numbers on these metal sheets and play the game with the help of a balance type indicator.

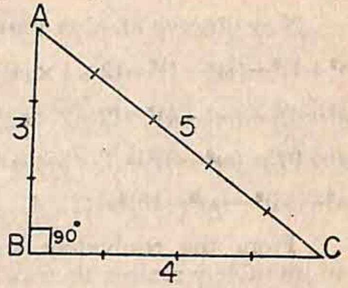
19. Develop different variants of a similar game as in problem 17 above in case of base three.

20. In the above games let each number be a code number for some statement that the player chooses from the given list. You can now read his mind with this variant of the game.

6

Squares of Numbers

6.1. It is said that the ancient Egyptians knew the art of constructing a right angle perfectly, by taking a thread measuring say 12 units of length, marking it and spreading it in the shape of triangle ABC as shown in the figure. The angle formed at B was always found to be 90°.



The famous mathematician Pythagoras later discovered the property of right angled triangles that the square on the hypotenuse is equal in area to the sum of the squares on the other two sides. In other words, in $\triangle ABC$, if $\angle B$ is a right angle,

then $$AC^2 = AB^2 + BC^2.$$

What we are interested to discuss here, is to find out the integral values for AB, BC and AC for which the above relation holds good. The simplest and widely known triple is (3, 4, 5) such that $5^2 = 3^2 + 4^2$. It is easy to see that :

$$5^2 = 3^2 + 4^2$$
$$\Leftrightarrow 5^2k^2 = 3^2k^2 + 4^2k^2, \qquad k \in N$$
$$\Leftrightarrow (5k)^2 = (3k)^2 + (4k)^2$$

Hence $(3k, 4k, 5k)$ is one such series of triples. ...(A)

We may write some of the triples from the above by assigning values 1, 2, 3,...to k so that we have :

$$(3, 4, 5), (6, 8, 10), (9, 12, 15), \ldots$$

Another series is generated by the triple (5, 12, 13) because $5^2 + 12^2 = 13^2$, so that another series would be generated by

$$(5k, 12k, 13k) \qquad \ldots(B)$$

Any triple satisfying the above property is known as a **Pythagorean triple**.

6.2. Let us examine the above question once again by observing the following pattern :

$$(1^2+1)^2-(1^2-1)^2=4 \quad =2^2 \quad \Leftrightarrow \quad 2^2-0^2=2^0$$

$$(2^2+1)^2-(2^2-1)^2=16 \quad =4^2 \quad \Leftrightarrow \quad 5^2-3^2=4^2$$

$$(3^2+1)^2-(3^2-1)^2=36 \quad =6^2 \quad \Leftrightarrow \quad 10^2-8^2=6^2$$

$$(4^2+1)^2-(4^2-1)^2=4\times 16=8^2 \quad \Leftrightarrow \quad 17^2-15^2=8^2$$

If we introduce a variable "m" for values 1, 2, 3, 4 in the above examples, we get $(m^2+1)^2-(m^2-1)^2=(2m)^2$, $m \in$ N which is a true statement.

Now observe another pattern :

$$(m^2+1)^2-(m^2-1)^2=(2\times 1\times m)^2 \Leftrightarrow (m^2+1^2)^2-(m^2-1^2)^2=(2\times 1\times m)^2$$

$$(m^2+4)^2-(m^2-4)^2=(2\times 2\times m)^2 \Leftrightarrow (m^2+2^2)^2-(m^2-2^2)^2=(2\times 2\times m)^2$$

$$(m^2+9)^2-(m^2-9)^2=(2\times 3\times m)^2 \Leftrightarrow (m^2+3^2)^2-(m^2-3^2)^2=(2\times 3\times m)^2$$

$$(m^2+16)^2-(m^2-16)^2=(2\times 4\times m)^2 \Leftrightarrow (m^2+4^2)^2-(m^2-4^2)^2=(2\times 4\times m)^2$$

From the equivalent statements listed on the right side, we can intuitively realise, that we may introduce another variable n, $n \in$ N in the above statements so that

$$(m^2+n^2)^2-(m^2-n^2)^2=(2mn)^2$$

This is a very important relation as it can generate all the Pythagorean number triple. Therefore, before we proceed further it is advisable to study this relation in some detail.

If m and n are both even or both odd then both m^2+n^2 and m^2-n^2 are both even and in fact contain 4 as a factor. Therefore the triple generated by it is a multiple of a triple found earlier in the series. An illustration will make the observation clear.

Let $m=4$, $n=2$

Then $m^2+n^2=20$, $m^2-n^2=12$, $2mn=16$

The triple is (20, 16, 12) which is four times the primitive triple (5, 4, 3).

Let $m=3$, $n=1$

Then $m^2+n^2=10$, $m^2-n^2=8$, $2mn=6$.

The triple is (10, 8, 6) which is twice the triple (5, 4, 3).

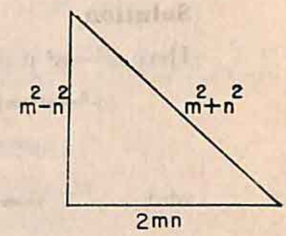
We need therefore consider the case when only one of m and n is odd, and, to make the matters simple, prime with respect to each other. (Why ?)

Let us have m odd and n even. (It does not matter if reverse is the case.)

Then $2mn$ is always even,

and (m^2+n^2) and (m^2-n^2) are both odd.

If we now refer back to a right triangle whose legs are $2mn$ and m^2-n^2 and the hypotenuse is m^2+n^2 we observe that



1. One of the two legs is odd.

2. The hypotenuse is always odd.

Giving m and n integral values satisfying the condition stated above one can generate a series of Pythagorean number triples, no two of which are similar *i.e.*, one cannot be obtained by multiplying the other by a constant factor.

m	n	*triple*
1	2	3, 4, 5
1	4	8, 15, 17
1	6	12, 35, 37
1	8	16, 63, 65
....	....	....
3	2	5, 12, 13
3	4	7, 24, 25
....	....	....
5	2	20, 21, 29

In the above table wherever $m > n$ numerical value of m^2-n^2 has been taken.

We are now in a position to consider variations of the Pythagorean Problem.

Problem 1. *Given the even leg of a primitive right triangle find the other two sides.*

Solution.

This is very simple. Given $2mn$ one has only to find m and n such that one and only one of m and n is odd and that m and n do not have a common factor. Then using the above table or by finding the sum and the difference of the squares of these numbers we get the other two sides.

Problem 2. *Given the odd leg of a primitive right triangle, find the other two sides.*

Solution.

Here m^2-n^2 is given. It is required to find m and n.

$$m^2-n^2=(m+n)(m-n)$$

$$=u.v \text{ say}$$

and $$m=\frac{u+v}{2}$$

and $$n=\frac{u-v}{2}$$

Since one and only one of m and n is odd, both u and v are odd. Thus the given number should be factorised into relatively prime factors and m and n found as indicated above.

Example.

m^2-n^2	$(m+n)(m-n)$	m	n	*Pythagorean triple*
3	3.1	2	1	3, 4, 5
5	5.1	3	2	5, 12, 13
7	7.1	4	3	7, 24, 25
15	15.1	8	7	15, 112, 113
15	5.3	4	1	8, 15, 17

Problem 3. *Given the hypotenuse of a right triangle find the legs of the triangle.*

Solution.

The problem reduces to expressing the given odd number (the length of the hypotenuse) as a sum of squares of two relatively prime numbers, atmost one out of which is odd. Following cases arise.

Case 1. The given number is a prime of the type $(4k+1)$ or contains a factor of the type $(4k+1)$

Example.

$$29=4.7+1$$

$$91=7.13$$

$$=7(4.3+1)$$

In the first case there is just one set of values (m, n).

In the second case a solution is not always possible. It is possible if and only if the number contains factors of the type $(4k-1)$ repeated even number of times.

e.g., $29=4.7+1=5^2+2^2$ only one solution

$97=7.13=(4.2-1)(4.3+1)$ no solution

$45=3^2.5=(4.1)-1]^2(4.1+1)=3^2+6^2$

Case 2. When the given number is a prime of the type, $(4k-1)$ or contains a factor of the type $(4k-1)$.

The solution is possible if and only if the factors of the type are repeated even number of times whether or not any factor not belonging to this class is present.

e.g. $385=5.7.11$

$=(4.1+1)(4.2-1)(4.3-1)$ no solution.

Note. If the number has factors of the form $(4k+1)$ only here can be a number of solutions.

e.g., $1105=5.13.17$

$$=(4.1+1)(4.3+1)(4.4+1)$$

$$=(2^2+1^2)(3^2+2^2)\{4^2+1^2\}$$

$$=(7^2+4^2)\{4^2+1^2\}$$

$$=(33^2+4^2)$$

$$=32^2+9^2$$

$$=24^2+23^2$$

The algebraic proof for this variety lies in the following identity.

$$Z=(a^2+b^2)(c^2+d^2)$$

$$=a^2c^2+b^2d^2+a^2d^2+b^2c^2$$

$$=(a^2c^2 \pm 2acbd+b^2d^2)+(a^2d^2 \mp 2abcd+b^2c^2)$$

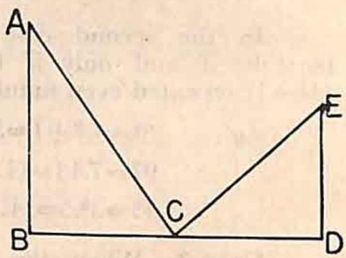
$$=(ac+bd)^2+(ad-bc)^2 \text{ or } (ac-bd)^2+(ad+bc)^2$$

Problem Set 6.1

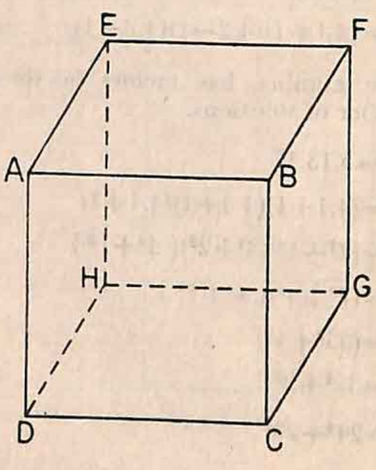
1. Show that if (m, n, p) is a pythagorean triple, then (km, kn, kp) is also a pythagorean triple.

2. Show that if (m, n, p) is an ordered pythagorean triple, such that $m^2=2n+1$ then $p=n+1$.

3. A ladder 17 m long rests against one wall of a room and its top reaches a height of 15 m from the ground. If the ladder is made to rest against the opposite wall without changing the point of contact with the ground (point C in the figure) its top reaches a height of 8 m only. Find the distance between the two walls.



4. If the dimensions of a cuboid be a, b, c respectively, find the length of the diagonal BH.



5. If $a^2+b^2=m^2$

$m^2+c^2=d^2$

then $a^2+b^2+c^2=d^2$

One set of integral values of (a, b, c, d) is $(3, 4, 12, 13)$.

Find three more such sets of integral values of (a, b, c, d) satisfying this relationship.

6. We have
$$7^2+24^2=25^2$$
$$32^2+24^2=40^2$$
$$143^2+24^2=145^2$$

(*a*) Find some more pythagorean triples in which 24 occurs as one of the numbers.

(*b*) Find all possible pythagorean triples in which one of the numbers is

(*i*) 8 (*ii*) 15 (*iii*) 36.

7. Show that there exists no pythagorean triple (x, y, z) such that $x=2$.

8. Show that for every odd number x greater than 1, there exists a pythagorean triple (x, y, z).

9. Show that in a pythagorean triple (x, y, z) if x and y are both not divisible by 3, 7 or 11 then z is also not divisible by 3, 7 or 11 respectively.

10. Prove the truth or falsity of the statement :

In any pythagorean triple (x, y, z)

(*i*) x and y both must not be odd.

(*ii*) x and y may be a pair of odd or even numbers.

(*iii*) x and y both may be even.

(*iv*) x and y both must be even.

(*v*) x and y must be a pair of odd or even numbers.

7

Clock Arithmetic

7.1. Consider an infinitely long straight line on which points have been marked off at unit intervals. If we associate with each of them a natural number taken in order we get an infinite supply of numbers that we need for our classical arithmetic.

In this chapter we are going to deal with a number system where the supply of numbers is limited. In terms of a number line if we turn the line with a limited numbers to form a circle on the one hand we will have a finite (limited) number of intervals and on the other our journey along this line need not come to an end because of its limited length.

Not only do we come across such systems in our every-day life, there is atleast one such system we use it *every moment of our life.* The number line on the face of a clock is a member of such a family. It is for this reason that the arithmetic where the supply of numbers is limited is called clock arithmetic. For reasons which would be obvious soon, this arithmetic is sometimes called Modular Arithmetic.

Whenever one looks at a clock (or watch on his wrist) he is seeking an answer to questions of the type :

(*a*) If it is 2 o'clock now and I work 3 hours what time will I finish ?

(*b*) If it is 8 o'clok now and if I have been waiting at the railway station since 2 o'clock, how long have I been waiting ?

(*e*) If it is 3 o'clock now and the train halts every two hours, what time will it be before the train reaches the fourth station ?

(*d*) What time should I give to each item of work if I start at 2 o'clock and I have three such items and I must finish by 11 o'clock ?

To seek answers to the above questions we carry out some mental calculations as indicated below :

(*a*) $2+3=5$

(*b*) $8-2=6$

(*c*) $4\times2=8$; $3+8=11$

(*d*) $11-2=9$; $9\div3=3$

The four basic operations shown above are identical with those of classical arithmetic. Let us have a further look at addition. If I started at 9 o'clock and worked for 7 hours, what time would I finish ? The answer is obviously 4 o'clock. Thus, when dealing with clocks the statement

$$9+7=4 \qquad \text{is true.}$$

Similarly $\qquad 3\times5=3$

is also true with clocks.

The addition and multiplication tables for numbers on the face of the clock are shown in tables I and II below :

+	1	2	3	4	5	6	7	8	9	10	11	12
1	2										12	1
2	3									12	1	2
3	4								12	1	2	3
4	5							12	1	2		4
5	6						12	1	2			5
6	7					12	1	2				6
7	8				12	1	2					7
8	9			12	1	2						8
9	10		12	1	2							9
10	11	12	1	2								10
11	12	1	2									11
12	1	2										12

Table I

Clock Addition Table

X	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8				
2	2	4	6	8	10		2	4	6			
3	3	6	9	12	3			12	3	6		
4	4	8	12	4	8				12	4	8	
5	5	10	3	8	1					2	7	12
6	6	12	6	12	6				6	12	6	
7	7	2	9	4	9							
8	8	4	12	8	4		8	4	12			
9	9	6	3	12	9							
10	10	8	6	4	2	12	10	8				
11	11	10	9	8	7		5	4	3			
12	12	12	12	12	12				12	12	12	

Table II

Clock Multiplication Table

These two tables possess the following striking features :

(*a*) The tables are limited in extent.

(*b*) No number beyond 12 is needed either as an addend or to express a sum as also either as a factor or to express a product.

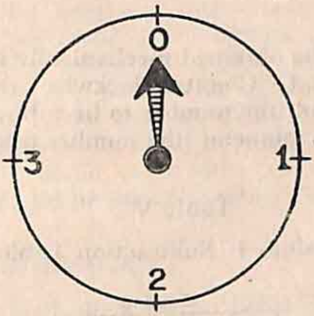
(*c*) There is no number 0 in these tables (12 can, however, be replaced by 0 without any difficulty).

(*d*) Since every addition and multiplication of two numbers in our clock arithmetic yield a unique number in the original set 1, 2, ..., 11, 12, we say this set of numbers is closed for the operations of addition and multiplication. Similar statement cannot be made for any *limited* or *finite* set of whole numbers in our usual arithmetic.

The system of arithmetic in which only the numbers 1, 2, ... 11, 12 are used is said to have a modulus of 12. As stated above 12 can be replaced by 0 without any loss of generality. Accordingly modular 4 (or modulo 4) system would contain just the four numbers 0, 1, 2, 3 ; a modulo 3 : 0, 1, 2 etc.

7.2. Basic Arithmetical Operations in other Modular Systems

We will restrict the numbers to four and study in greater detail a modulo system. This system will contain the numbers 0, 1, 2, 3.



As with the clock, let the movement of the hand "clockwise" imply addition. Let us, with this speculation, compute the modulo 4 addition table and the multiplication table.

Addition and Multiplication

Table III

Modulo 4 Addition Table

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Table IV

Modulo 4 Multiplication Table

X	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

It would be seen from the tables that

(*a*) Addition and multiplication are commutative.

$1+2=3=2+1$

$2\times3=2=3\times2$

(*b*) Addition and multiplication are associative.

$1+2+3=(1+2)+3=1+(2+3)=2$

$1\times2\times3=(1\times2)\times3=1\times(2\times3)=2$

(*c*) Multiplication is distributive with respect to addition.

$2\times(3+1)=2\times 0=0$

$2\times(3+1)=(2\times 3)+(2\times 1)=2-2=0$

(*d*) This system possesses identity elements with respect to both the operations.

Subtraction

Subtraction can be obtained mechanically in the modular system by moving the hand "Counter-clockwise" through the intervals given by the subtrahend (the number to be subtracted) starting from the figure given by the minuend (the number from which to be subtracted).

Table V

Modulo 4 Subtraction Table

Minuend

−	0	1	2	3
0	0	1	2	3
1	3	0	1	2
2	2	3	0	1
3	1	2	3	0

Subtrahend

It would be seen from the modulo 4 Subtraction table (table V) that though with our usual set of non-negative numbers in classical arithmetic subtractions like $1-2$ or $2-3$ have no results, we have no such problems in our modular system. The remainder for $1-2$ would be found by starting with the pointer at 1 (the minuend) and moving counter-clockwise 2 (the subtrahend) points, 0, 3 to indicate 3 as the remainder for $1-2$.

The following properties of this system with respect to the operation of subtraction can be directly verified from the table above.

(*a*) The system is "closed" for subtraction.

(*b*) The operation is not commutative.

(*c*) The subtraction is not associative.

(*d*) Multiplication is distributive over subtraction.

Division

As in classical arithmetic the concept of division can be understood by two different approaches. Firstly, the division may be

thought of as an operation which is inverse of multiplication, *i.e.*, as an operation for finding the value of a missing factor in a multiplication. For example $2 \div 3 = x$ is equivalent to $3 \times x = 2$.

Stated in words this statement is a question :

What number multiplied by 3 yields 2 as the product ?

Alternatively, the division may be conceived as repeated subtraction until the dividend is exhausted (leaving a remainder zero). Here the division differs from that in the classical arithmetic where the repeated subtraction process is terminated, the moment the remainder is less than the divisor. In modular arithmetic either the division is exact or there is no quotient. This additional restriction should cause some difficulty in preparing the division tables. It should be borne in mind that division by zero is excluded.

Let us answer the question :

What number multiplied by 2 yields 2 as the product in a modulo 4 system ?

Looking back at Table IV we see that

$$2 \times 1 = 2$$

and $$3 \times 2 = 2$$

Thus there are two valid answers to this question. Therefore $2 \div 2$ has no unique answer. Hence in view of the closure property in this system $2 \div 2$ has no quotient. We do not come across such cases with prime modulo systems (See Tables VI and VII).

Table VI

Modulo 3 Division Table

DIVISOR	÷	0	1	2
	0	–	–	–
	1	0	1	2
	2	0	2	1

Table VII

Modulo 4 Division Table

DIVISOR	÷	0	1	2	3
	0	–	–	–	–
	1	0	1	2	3
	2	–	–	–	–
	3	0	3	2	1

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